MATH2048 24-25 Midterm 2 Solution

1(a). $\beta = \{e_i\}$.

$\Rightarrow [T_2]_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$f_T(t) = t^2 - 1 = (t+1)(t-1)$

Let $\lambda_1 = 1$, $\lambda_2 = -1$.

$E_{\lambda_1} = N\left(\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\right) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

$E_{\lambda_2} = N\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$.

$\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) = 2 = \dim(\mathbb{R}^2)$

$\therefore$ T_2 diagonalizable. $\mu(\lambda_i) = \gamma(\lambda_i)\ \forall i$

(b). Suppose $(x_1, \ldots, x_n)$ is an eigenvector with eigenvalue λ.

Then $x_{i+1} = \lambda x_i \quad \forall i$

$\Rightarrow x_i = \lambda^{i-1} x_1$ and $\lambda^i = 1$

Then $E_\lambda = \text{span}\{(1, \lambda, \lambda^2, \ldots, \lambda^{n-1})\}$

with dimension 1, and

$\lambda^n = 1$ implies $\lambda = 1$ or -1.

So $\sum_i \mu(\lambda_i) \leq 2 \cdot 1 = 2 < n$. for $n \geq 3$.

$\therefore$ T_n not diagonalizable for $n \geq 3$.

2(a). $\beta = \{1, x, x^2\}$.

$$[T]_\beta = \begin{pmatrix} -1 & 0 & 0 \\ 3 & 3 & -1 \\ 3 & 1 & 1 \end{pmatrix}$$

$$f_T(t) = (-1-t)((3-t)(1-t)+1)$$
$$= -t^3 + 3t^2 - 4$$

By Cayley-Hamilton Thm, $f_T(T) = T_0$

$\therefore -T^3 + 3T^2 - 4I = T_0 \Rightarrow T^3 = 3T^2 - 4I$.

$\therefore f(x) = 3x^2 - 4$.

(b). $[g]_\beta = (0, 3, 2)^T$.

$$[Tg]_\beta = \begin{pmatrix} -1 & 0 & 0 \\ 3 & 3 & -1 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 5 \end{pmatrix}$$

$$[T^2g]_\beta = \begin{pmatrix} -1 & 0 & 0 \\ 3 & 3 & -1 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 16 \\ 12 \end{pmatrix}$$

Note that $[g]_\beta$, $[Tg]_\beta$ lin. ind. but

$T^2g = 4Tg - 4g \Rightarrow W = \text{span}\{g, Tg\}$.

$\Rightarrow \dim(W) = 2$

$f_{T_W}(t) = t^2 - 4t + 4$.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, "f"'] & R(T) \subseteq W \arrow[d, "g"] \\ & F \end{tikzcd}$$

3. ($\Rightarrow$) Suppose T^* onto. Then $\forall f \in V^*$,
$\exists\ g \in W^*$ s.t. $T^*(g) = f \Rightarrow gT = f$.
Pick $x \in N(T)$, $gT(x) = f(x) \Rightarrow f(x) = 0\ \forall f$
$\Rightarrow x = 0$.

$\therefore T$ one-to-one.

($\Leftarrow$) Suppose T one-to-one.
Let β be basis for $V \Rightarrow T(\beta)$ lin ind.
Extend β to γ as a basis for W.

Pick $f \in V^*$, define $g \in W^*$ as

$$g(w) = \begin{cases} f \circ T^{-1}(w) & \text{if } w \in T(\beta) \\ 0 & \text{if } w \in \gamma \setminus T(\beta). \end{cases}$$

Then $\forall v \in \beta$

$$\begin{aligned} T^*(g)(v) &= g(Tv) \\ &= f \circ T^{-1}(Tv) \\ &= f(v). \end{aligned}$$

$\Rightarrow T^* g = f. \Rightarrow T^*$ onto

4. Since U is diagonalizable.

Let $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$,

where E_{λ_i} are eigenspaces of U

Pick $x \in E_{\lambda_i}$

$$UT(x) = TU(x)$$

$$UT(x) = \lambda_i T(x) \Rightarrow T(x) \in E_{\lambda_i}$$

$$\Rightarrow E_{\lambda_i} \text{ is } T\text{-invariant } \forall i$$

Let $W_i = E_{\lambda_i}$, then (*).

$V = W_1 \oplus \cdots \oplus W_k$ is a direct sum decomposition of V into T-invariant subspaces.

Since T is diagonalizable, let $\{v_1, \ldots, v_n\}$ be eigenbasis of T for V.

By (*), for all v_j $\exists!$ w_{ij} $\forall i, j$ s.t.

$$v_j = \sum_{i=1}^{k} w_{ij} \quad \forall j, \quad w_{ij} \in W_i \quad \forall i.$$

$$\Rightarrow T(v_j) = \sum_i T(w_{ij}) \quad \forall j$$

$$\lambda_j v_j = \sum_i T(w_{ij}) \quad \forall j$$

$$\sum_i \lambda_j w_{ij} = \sum_i T(w_{ij}) \quad \forall j$$

$$\lambda_j w_{ij} = T(w_{ij}) \quad \forall i, j \quad \text{by } (*).$$

$\therefore$ $w_{ij} = 0$ or

w_{ij} is an eigenvector of T with eigenvalue λ_j.

Note that $W_i = \text{span}\{w_{i1}, \dots, w_{in}\}$ $\forall i$

(this set contains only zero vectors or eigenvectors)

Removing zero vectors, we obtain a basis β_i of eigenvectors of T for W_i $\forall i$

Note that β_i also contains eigenvectors of U only. $\forall i$.

So both $[U_{W_i}]_{\beta_i}$ and $[T_{W_i}]_{\beta_i}$ diagonal $\forall i$

Let $\beta = \bigcup \beta_i$, then $[U]_\beta$, $[T]_\beta$ diagonal

$\Rightarrow$ $[UT]_\beta = [U]_\beta [T]_\beta$ diagonal.

$\Rightarrow$ UT is diagonalizable.

$$V = W_1 \oplus \cdots \oplus W_k \quad \begin{array}{ccc} \{v_1, & \dots, & v_n\} \\ \downarrow & & \downarrow \\ w_1 & & w_{1n} \\ + & & + \\ w_2 & & w_{2n} \\ + & & + \\ \vdots & & \vdots \\ + & & + \\ w_k & & w_{kn} \end{array}$$

5(a). Suppose $Tv = \lambda v$, $v \neq 0$.

Then easy to see $T^m v = \lambda^m v$ by MI

$\Rightarrow \lambda^m$ is an eigenvalue of T^m $\forall\, m \geq 2$.

(b). $\lambda = 0$ is obvious. So assume $\lambda \neq 0$.

Suppose λ^m is an eigenvalue of T^m $\forall\, m \geq 2$,

but λ is not an eigenvalue of T.

Let (p_i) be a strictly increasing seq of primes,

Pick $v_i \in V$ s.t. $T^{p_i} v_i = \lambda^{p_i} v_i$.

Observe that $T^{ap_i} v_i = \lambda^{ap_i} v_i$ by (a). — ①

<u>Claim</u>: $\{v_1, \ldots, v_k\}$ lin. ind. $\forall\, k \in \mathbb{N}$.

<u>Proof</u>: $k = 1$ true.

Suppose $k = n$ true, $k = n+1$ not true.

Then $v_{n+1} = \sum_{i=1}^{n} a_i v_i$ — ②

Let $M = \prod_{i=1}^{n} p_i$, then $T^M v_i = \lambda^M v_i$ $\forall i$ by ①

$\Rightarrow T^M v_{n+1} = \lambda^M v_{n+1}$ by ②

Since M, p_{n+1} coprime, $\exists$ a, b s.t.

$aM - bp_{n+1} = 1$

Note that $T^{bp_{n+1}} v_{n+1} = \lambda^{bp_{n+1}} v_{n+1}$

We have $\begin{cases} T^{aM} v_{n+1} = \lambda^{aM} v_{n+1} \\ T^{bp_{n+1}} v_{n+1} = \lambda^{bp_{n+1}} v_{n+1} \quad (\neq 0) \end{cases}$

$\Rightarrow T\left(T^{bp_{n+1}} v_{n+1}\right) = \lambda \left(T^{bp_{n+1}} v_{n+1}\right)$

$\Rightarrow \lambda$ is an eigenvalue of T. $\therefore$ Contradiction

So by MI, the claim is proved.

However, it is not possible to have $n+1$ lin. ind. vectors when $\dim(V) = n$. $\Rightarrow$ Contradiction

$\therefore$ The converse of (a) is proved.

(c). Let $V = \mathbb{R} \oplus \mathbb{R} \oplus \ldots$

$w_n = e^{2\pi i/n}$, $\quad w_n^n = 1$

$T(e_i) = w_{i+1} e_i$

Then $T^m e_{m-1} = e_{m-1} = \lambda^m e_{m-1}$. for $\lambda = 1$

But $\lambda = 1$ is not an eigenvalue of T as $\nexists\, v \in V$ s.t. $T(v) = v$.

$\therefore$ The statement is disproved.

$$T\left(T^{by}\overset{\lambda^{k} v_2}{v_1}\right) = \lambda\left(\lambda^{by}\overset{\lambda^{k} v_2}{v_1}\right)$$

$$T^{by+1} v_1 = \lambda^{by+1} v_1$$

$$T^{x} v_1 = \lambda^{x} v_1 \quad \Rightarrow \quad T^{ax} v_1 = \lambda^{ax} v_1$$

$$T^{y} v_2 = \lambda^{y} v_2 \quad \Rightarrow \quad T^{by} v_2 = \lambda^{by} v_2$$